De figuris quadrangulis, circum quas vel in quas circulus perscribi potest.

Annis 1846 et 1850 in programmatis Progymnasii Roesseliensis aequationes tetragonometricas typis imprimendas curavi, ut eas ad nonnulla systemata problematum tetragonometricorum adhiberem. Illuc igitur paragraphi spectant, quae in hoc libello commemoratae sunt. Notae, quibus usus sum, hae sunt: si angulos figurae eo ordine, quo se subsequuntur, literis $\alpha, \beta, \gamma, \delta$ signavisti, literae a, b, c, d denotant latera $\alpha\beta$, $\beta\gamma$, $\gamma\delta$, $\delta\alpha$ et literae e, e^1 lineas diagonales $\alpha\gamma$, $\beta\delta$. Ad has accedunt: r, ϱ radii circulorum, qui circum figuram et in figuram perscribi possunt, ϱ_1 radius circuli, qui latus a et producta latera b, d tangit, et eodem modo ϱ_2, ϱ_3, ϱ_4 radii circulorum, qui latus b, c, d et producta latera finitima tangunt.

A. Figurae quadrangulae, circum quas circulus perscribi potest.

1) Quum in hujusmodi figuris summae angulorum oppositorum aequales sint ($\alpha+\gamma=\beta+\delta$), quattuor quantitatibus definiuntur. Si igitur latera et anguli figurae locum quantitatum definientium tenent, summum duo anguli dati esse possunt iique sibi finitimi sint oportet. Quamobrem data esse possunt:

1) a, b, c, d.	3) a, b, α, β.
2) a, b, c, β.	4) a, c, α, β.

Aequatio $\alpha+\gamma=\beta+\delta$ non patitur, has figuras quandrangulas esse secundi generis.

2) Problema 1. Datis 4 lateribus $(a, b, c, d,)$, quaerantur anguli $(\alpha, \beta, \gamma, \delta)$, area (J), radius circuli (r).

Solutio. Datis quantitabus 2 figurae quadrangulae satisfaciunt, una primi, altera tertii generis.

Quod attinet ad priorem, ex §. 12. sequitur

$$\sin\frac{\alpha}{2} = \sqrt{\frac{[(b+c)+(a-d)]\,[(b+c)-(a-d)]}{4\,(bc+ad)}}$$

$$\sin\frac{\beta}{2} = \sqrt{\frac{[(c+d)+(a-b)]\,[(c+d)-(a-b)]}{4\,(cd+ab)}}$$

$$\gamma = 180^\circ - \alpha, \text{ vel } \sin\frac{\gamma}{2} = \sqrt{\frac{[(a+d)+(b-c)]\,[(a+d)-(b-c)]}{4\,(ad+bc)}}$$

$$\delta = 180^\circ - \beta, \text{ vel } \sin\frac{\delta}{2} = \sqrt{\frac{[(a+b)+(c-d)]\,[(a+b)-(c-d)]}{4\,(ab+cd)}}$$

Area figurae $J = \frac{1}{2} ab \sin\beta + \frac{1}{2} cd \sin\delta$. Quum vero hic $\sin\beta = \sin\delta$, $J = \frac{1}{2}(ab + cd)\sin\beta$ $= (ab + cd) \sin\frac{\beta}{2} \cos\frac{\beta}{2} = (ab+cd) \sin\frac{\beta}{2} \sin\frac{\delta}{2}$. Igitur

$$J = \frac{1}{4}\sqrt{(a+b+c-d)(a+b-c+d)(a-b+c+d)(-a+b+c+d)}$$

sive, posito $a + b + c + d = 2s$

$$J = \sqrt{(s-a)(s-b)(s-c)(s-d)}$$

Circulus circum figuram quadrangulam perscriptus idem est, qui circum triangulum $\alpha\beta\gamma$ perscribi potest. In hoc vero est $r = \frac{e}{2\sin\beta}$ et

$$e^2 = (a-b)^2 + 4ab\sin^2\frac{\beta}{2} = (a-b)^2 + \frac{ab(c+d+a-b)(c+d-a+b)}{ab+cd}$$

$$= \frac{(ac+bd)(bc+ad)}{ab+cd}.$$ Quum praeterea $\sin\beta = \frac{2J}{ab+cd}$,

$$r = \frac{1}{4J}\sqrt{(ab+cd)(ac+bd)(bc+ad)}, \text{ sive}$$

$$r = \sqrt{\frac{(ab+cd)(ac+bd)(bc+ad)}{(a+b+c-d)(a+b-c+d)(a-b+c+d)(-a+b+c+d)}}$$

Quod attinet ad figuram quadrangulam tertii generis, ex §. 34. sequitur:

$$\sin\frac{\alpha}{2} = \sin\frac{\gamma}{2} = \sqrt{\frac{[(a-d)+(b-c)]\,[(a-d)-(b-c)]}{4(bc-ad)}}$$

$$\sin\frac{\beta}{2} = \sin\frac{\delta}{2} = \sqrt{\frac{[(a-b)+(c-d)]\,[(a-b)-(c-d)]}{4(cd-ab)}}$$

Quaeri potest, qui sit valor (liceat mihi hoc vocabulo uti) harum quantitatum, si $bc-ad=0$. Ex aequatione $a:b = c:d$, i. e. $bc - ad = 0$, sequitur $\triangle\,\alpha\beta\gamma \sim \triangle\,\alpha\gamma\delta$, et quum latus $\alpha\gamma$ iis commune sit, etiam $\triangle\,\alpha\beta\gamma \cong \alpha\gamma\delta$; igitur $a = c$, et $b = d$. Si vero $b = d$, (unde sequitur, $a \parallel c$),

$$\sin\frac{\alpha}{2} = \sqrt{\frac{(a-c)(a+c-2b)}{4b(c-a)}} = \sqrt{\frac{2b-a-c}{4b}},$$

igitur $\cos\alpha = 1 - 2\sin^2\frac{\alpha}{2} = \frac{a+c}{2b} = \cos\beta$.

Quoniam praeterea $a = c$, $\cos\alpha = \cos\beta = \frac{a}{b}$, quod idem ex eo sequitur, quod, posita aequatione $ad = bc$, b et d diametri circuli circum figuram perscripti sunt.

Quum in hujusmodi figuris $\sin\beta = -\sin\delta$, area $J = {}^1/_2\,(ab - cd)\sin\beta$. Est vero

$$\sin\beta = \frac{1}{2(cd-ab)}\sqrt{[(c+d)+(a+b)]\,[(c+d)-(a+b)]\,[(a-b)+(c-d)]\,[(a-b)-(c-d)]};$$

$$\text{igitur } J = \frac{1}{4}\sqrt{[(c+d)+(a+b)]\ [(c+d)-(a+b)]\ [(a-b)+(c-d)]\ [(a-b)-(c-d)]}$$

$$= \sqrt{s\,[s-(a+d)]\ [s-(b+d)]\ [s-(c+d)]}$$

$$r = \sqrt{\frac{(ab-cd)\ (ac-bd)\ (bc-ad)}{s\,[s-(a+d)]\ [s-(b+d)]\ [s-(c+d)]}}.$$

3) Problema 2. Datis 3 lateribus (a, b, c) et 1 angulo (β), quaerantur quartum latus (d), anguli non dati (α, γ, δ), area (J); radius circuli (r).

Solutio. Si $\beta < 180^0$, secundum §. 10

$$0 = a \sin \alpha - b \sin (\alpha + \beta) - c \sin \beta,$$

$$\text{unde sequitur: } \frac{a - b \cos \beta}{b \sin \beta} \sin \alpha - \cos \alpha = \frac{c}{b}$$

$$\text{et } \sin (\alpha - \varphi) = \frac{c}{b} \sin \varphi, \quad \text{si } \cot \varphi = \frac{a - b \cos \beta}{b \sin \beta}.$$

Quum forma $\cot \varphi = \frac{a - b \cos \beta}{b \sin \beta}$ ad calculum logarithmicum minus apta sit, quaeras angulos $\angle (ae)$, $\angle (be)$, utens aequationibus $\operatorname{tang} [\angle (ae) - \angle (be)] = \frac{b+a}{b-a} \cot \beta$ et $\angle (ae) + \angle (be) = 180^0 - \beta$. Quum porro $a : e = \sin \angle (be) : \sin \beta$ et $e : c = \sin \delta : \sin \angle (\delta e) = \sin \beta : \sin \angle (de)$, $a : c = \sin \angle (be) : \sin \angle (de)$, ergo $\sin \angle (de) = \frac{c \sin \angle (be)}{a}$.

Angulus $\angle (de)$ inde 2 valores sibi assumere potest. Si $\beta > 90^0$ et $c > \sqrt{a^2+b^2-2ab\cos\beta}$, 2 figurae quadrangulae primi generis exoriuntur. Si $\beta > 90^0$ et $c < \sqrt{a^2+b^2-2ab\cos\beta}$, aut si $\beta < 90^0$ et $c < \sqrt{a^2+b^2-2ab\cos\beta}$, 1 figura quadrangula primi et 1 tertii generis exoritur. Si postremo $\beta < 90^0$ et $c > \sqrt{a^2+b^2-2ab\cos\beta}$, 2 figurae quadrangulae tertii generis exoriuntur.

In figuris quadrangulis primi generis est $\alpha = \angle (ae) + \angle (de)$, in figuris quadrangulis tertii generis $\alpha = \angle (ae) - \angle (de)$.

Latus quartum figurarum quadrangularum primi generis ex §. 11. invenitur:

$$d = -c \cos \beta \pm \sqrt{a^2 + b^2 - 2ab \cos \beta - c^2 \sin^2 \beta},$$

unde apparet, $c \sin \beta < \sqrt{a^2+b^2-2ab\cos\beta}$ esse debere. Quum autem haec forma ad calculum logarithmicum minus apta sit et quum anguli $\angle (ce)$ et $\angle (de)$ jam inventi sint, d facilius computatur aequatione

$$d = \frac{c \sin \angle (ce)}{\sin \angle (de)}$$

$$J = \tfrac{1}{2}\,(ab + cd) \sin \beta,$$

ubi in locum lateris d valor modo repertus substitui debet.

Aequationes, quae quantitates quaesitas figurarum quadrangularum tertii generis definiunt, eodem modo ex §§. 33, 34. deducuntur.

Radius circuli jam 3 quantitatibus a, b, β definitur. Secundum No. 2. enim est

$$r = \frac{e}{2\sin\beta} = \frac{1}{2\sin\beta}\sqrt{(a-b)^2 + 4ab\sin^2\frac{\beta}{2}}$$

$$= \frac{a-b}{2\sin\beta\cos\varphi}, \text{ si } \operatorname{tang}\varphi = \frac{2\sin\frac{\beta}{2}}{a-b}\sqrt{ab}.$$

Si $\beta > 180^\circ$, eodem modo 2 figurae quadrangulae tertii generis inveniuntur.

4) Problema 3. Datis 2 lateribus finitimis (a, b) et 2 angulis finitimis (α, β), quaerantur latera non data (c, d), area (J), radius circuli (r).

Solutio. Sit et α et $\beta < 180^\circ$. Quaerantur anguli $\angle(ae)$ et $\angle(be)$ aequationibus $\operatorname{tang}[\angle(ae) - \angle(be)] = \frac{b+a}{b-a}\cot\beta$ et $\angle(ae) + \angle(be) = 180^\circ - \beta$. Si tum $\alpha > \angle(ae)$, figura quadrangula primi generis oritur; si autem $\alpha < \angle(ae)$, tertii generis.

Quod attinet ad figuram quadrangulam primi generis, ex §. 10. invenitur:

$$c = \frac{a\sin\alpha - b\sin(\alpha+\beta)}{\sin\beta} = \frac{a\sin\alpha\cos 2\varphi}{\sin\beta\cos^2\varphi}, \text{ si } \operatorname{tang}\varphi = \sqrt{\frac{b\sin(\alpha+\beta)}{a\sin\alpha}},$$

$$d = \frac{a\sin(\beta-\alpha) + b\sin\alpha}{\sin\beta} = \frac{b\sin\alpha}{\sin\beta\cos^2\omega}, \text{ si } \operatorname{tang}\omega = \sqrt{\frac{a\sin(\beta-\alpha)}{b\sin\alpha}}.$$

Porro $J = \frac{1}{2}(ad + bc)\sin\alpha$, et si in locum laterum c et d valores inventi substituuntur,

$$J = \frac{ab\sin^2\alpha}{\sin\beta} + \frac{a^2\sin\alpha\sin(\beta-\alpha)}{2\sin\beta} - \frac{b^2\sin\alpha\sin(\alpha+\beta)}{2\sin\beta}.$$

Valor radii r idem est atque in problemate praecedente.

Eodem modo ex §§. 33. et 34. aequationes evolvuntur, quae de figura quadrangula tertii generis valent, si $\alpha < \angle(ae)$, aut si unus angulorum α et β, aut si uterque $> 180^\circ$.

5) Problema 4. Datis 2 lateribus oppositis (a, c) et 2 angulis finitimis (α, β), quaerantur latera non data (b, d), area (J), et radius circuli (r).

Solutio. Si non solum α et $\beta < 180^\circ$, sed etiam $\alpha + \beta < 180^\circ$, 1 figura quadrangula primi, et 1 tertii generis problemati convenit; si autem $\alpha + \beta > 180^\circ$, 1 tantum primi generis.

Quod attinet ad figuram quadrangulam primi generis, ex §. 10. invenitur:

$$b = \frac{a\sin\alpha - c\sin\beta}{\sin(\alpha+\beta)} = \frac{a\sin\alpha\cos 2\varphi}{\sin(\alpha+\beta)\cos^2\varphi}, \text{ si } \operatorname{tang}\varphi = \sqrt{\frac{c\sin\beta}{a\sin\alpha}},$$

$$d = \frac{a\sin\beta - c\sin\alpha}{\sin(\alpha+\beta)} = \frac{a\sin\beta\cos 2\omega}{\sin(\alpha+\beta)\cos^2\omega}, \text{ si } \operatorname{tang}\omega = \sqrt{\frac{c\sin\alpha}{a\sin\beta}}.$$

Quum quantitates $a \sin\alpha - c \sin\beta$ et $a \sin\beta - c \sin\alpha$ simul aut positivae aut negativae sint, prout $\alpha+\beta < 180^\circ$ aut $\alpha+\beta > 180^\circ$, $(a^2+c^2) \sin\alpha \sin\beta > ac\,(\sin^2\alpha + \sin^2\beta)$ esse debet.

Postquam in aequatione $J = {}^1/_2\,(ab + cd) \sin\beta$ in locum luterum b et d valores supra reperti substituti sunt, fit

$$J = \frac{(a+c)\,(a-c)\,\sin\alpha\,\sin\beta}{2\sin\,(\alpha+\beta)}$$

Secundum Nro. 2. est $r = \frac{1}{2\sin\beta}\sqrt{a^2+b^2-2ab\cos\beta}$. In hoc igitur casu fit

$$r = \frac{1}{2\sin(\alpha+\beta)}\sqrt{a^2+c^2+2ac\cos(\alpha+\beta)} = \frac{1}{2\sin(\alpha+\beta)}\sqrt{(a+c)^2-4ac\sin^2\frac{\alpha+\beta}{2}}$$

$$= \frac{(a+c)\sin\psi}{2\sin(\alpha+\beta)},\ \text{si}\ \cos\psi = \frac{2\sin\frac{\alpha+\beta}{2}}{a+c}\sqrt{ac}$$

Aequationes figuram quadrangulam tertii generis definientes, secundum §. 33. evolvuntur.

$$b = \frac{a\sin\alpha + c\sin\beta}{\sin(\alpha+\beta)} = \frac{a\sin\alpha}{\sin(\alpha+\beta)\cos^2\varphi}$$

$$d = \frac{a\sin\beta + c\sin\alpha}{\sin(\alpha+\beta)} = \frac{a\sin\beta}{\sin(\alpha+\beta)\cos^2\omega}$$

$$J = \frac{1}{2}(ab-cd)\sin\beta = \frac{(a+c)\,(a-c)\sin\alpha\sin\beta}{2\sin(\alpha+\beta)}.$$

$$r = \frac{1}{2\sin\beta}\sqrt{a^2+b^2-2ab\cos\beta} = \frac{1}{2\sin(\alpha+\beta)}\sqrt{a^2+c^2-2ac\cos(\alpha+\beta)}$$

$$= \frac{(a+c)\sin\psi}{2\sin(\alpha+\beta)},\ \text{si}\ \cos\psi = \frac{2\cos\frac{\alpha+\beta}{2}}{a+c}.$$

B. Figurae quadrangulae, in quas circulus perscribi potest.

a) Figurae quadrangulae, in quibus summae laterum oppositorum aequales sunt $(a+c = b+d)$.

6) Propter conditionem $a+c = b+d$ hae figurae 4 quantitatibus definiuntur, inter quas summum 3 latera. Data esse igitur possunt.

1) a, b, c, β.	4) a, b, α, γ.	7) a, c, α, δ.
2) a, b, α, β.	5) a, b, β, δ.	8) a, c, α, γ.
3) a, b, α, δ.	6) a, c, α, β.	9) a, α, β, γ.

Eadem conditio non patitur, has figuras quandrangulas esse tertii generis.

7) Problema 1. Datis 3 lateribus (a, b, c) et 1 angulo (β), quaerantur quartum latus (d), anguli non dati (α, γ, δ), area (J), radius circuli (ϱ).

Solutio. $d = a + c - b$.

Ex §. 16. sequitur $\sin\frac{\delta}{2} = \sin\frac{\beta}{2}\sqrt{\frac{ab}{c(a+c-b)}}$

Secundum §. 17. est $ac\sin^2\frac{\beta+\gamma}{2} = ab\sin^2\frac{\beta}{2} + bc\sin^2\frac{\gamma}{2}$,

unde sequitur $\sin\gamma + \frac{b - a\cos\beta}{a\sin\beta}\cos\gamma = \frac{ab+bc-ab\cos\beta}{ac\sin\beta}$.

Si hic ponitur $\operatorname{tang}\varphi = \frac{b-a\cos\beta}{a\sin\beta}$, invenitur $\sin(\gamma+\varphi) = \frac{ab+bc-ab\cos\beta}{ac\sin\beta}\cos\varphi$.

Quum vero haec forma ad calculum logarithmicum minus apta sit, quaerantur anguli $\angle(ae)$ et $\angle(be)$ aequationibus $\operatorname{tang}\frac{\angle(ae)-\angle(be)}{2} = \frac{b+a}{b-a}\cot\frac{\beta}{2}$ et $\angle(ae)+\angle(be) = 180^\circ - \beta$; porro anguli $\angle(ce)$ et $\angle(de)$ aequationibus $\operatorname{tang}\frac{\angle(ce)-\angle(de)}{2} = \frac{d+c}{d-c}\cot\frac{\delta}{2}$ et $\angle(ce)+\angle(de) = 180^\circ - \delta$. Si tum $\beta < 180^\circ$ et $\angle(de) < \angle(ae)$ 1 figura quadrangula primi generis oritur, in qua $\alpha = \angle(ae)+\angle(de)$ et $\gamma = \angle(be)+\angle(ce)$, atque 1 figura quadrangula secundi generis, in qua $\alpha = \angle(ae)-\angle(de)$ et $\gamma = \angle(be)-\angle(ce)$. Ad illam pertinet $\frac{1}{2}\delta < 90^\circ$, ad hanc $\frac{1}{2}\delta > 90^\circ$.

Si $\beta > 180^\circ$, 1 figura quadrangula secundi generis oritur, in qua $\frac{1}{2}\delta < 90^\circ$.

$$J = \frac{1}{2}ab\sin\beta + \frac{1}{2}cd\sin\delta. \text{ Est vero } \sin\frac{\delta}{2} = \sin\frac{\beta}{2}\sqrt{\frac{ab}{cd}},$$

$$\text{igitur } \cos\frac{\delta}{2} = \pm\sqrt{\frac{cd - ab\sin^2\frac{\beta}{2}}{cd}} \text{ et } \sin\delta = \pm\frac{2\sin\frac{\beta}{2}}{cd}\sqrt{abcd - a^2b^2\sin^2\frac{\beta}{2}}.$$

$$\text{Ergo } J = \frac{1}{2}ab\sin\beta \pm \sin\frac{\beta}{2}\sqrt{abc(a-b+c) - a^2b^2\sin^2\frac{\beta}{2}}$$

$$= \frac{ab\sin\frac{\beta}{2}\sin(\varphi \pm \frac{\beta}{2})}{\sin\varphi}, \text{ si } \sin\varphi = \sin\frac{\beta}{2}\sqrt{\frac{ab}{c(a-b+c)}}$$

Signum superius congruit cum $\delta < 180^\circ$, inferius cum $\delta > 180^\circ$.

Quum $J = \frac{1}{2}(a+b+c+d)\varrho$, hic igitur propter conditionem $a+c = b+d$, $J = (a+c)\varrho$,

$$\varrho = \frac{J}{a+c} = \frac{\frac{1}{2}ab\sin\beta + \sin\frac{\beta}{2}\sqrt{abc(a+c-b) - a^2b^2\sin^2\frac{\beta}{2}}}{a+c} = \frac{ab\sin\frac{\beta}{2}\sin(\varphi \pm \frac{\beta}{2})}{(a+c)\sin\varphi}$$

8) **Problema 2.** Datis 2 lateribus finitimis (a, b) et 2 angulis, qui ad unum horum laterum adjacent (α, β), quaerantur latera non data (c, d), anguli non dati (γ, δ), area (J), radius circuli (ϱ).

Solutio. Secundum §. 17. est $bd \sin^2 \frac{\alpha+\beta}{2} = ad \sin^2 \frac{\alpha}{2} + ab \sin^2 \frac{\beta}{2}$, igitur

$$d = \frac{ab \sin^2 \frac{\beta}{2}}{b \sin^2 \frac{\alpha+\beta}{2} - a \sin^2 \frac{\alpha}{2}} = \frac{a \sin^2 \frac{\beta}{2} \cos^2 \varphi}{\sin^2 \frac{\alpha+\beta}{2} \cos^2 \varphi}, \text{ si } \operatorname{tang} \varphi = \frac{\sin \frac{\alpha}{2}}{\sin \frac{\alpha+\beta}{2}} \sqrt{\frac{a}{b}}.$$

$$c = b - a + d.$$

Secundum §. 14. est $a \sin \frac{\alpha}{2} \sin \frac{\beta+\gamma}{2} = b \sin \frac{\gamma}{2} \sin \frac{\alpha+\beta}{2}$, unde sequitur

$$\cot \frac{\gamma}{2} = \frac{b \sin \frac{\alpha+\beta}{2} - a \sin \frac{\alpha}{2} \cos \frac{\beta}{2}}{a \sin \frac{\alpha}{2} \sin \frac{\beta}{2}} = \frac{\sin (\frac{\beta}{2} - \varphi)}{\sin \frac{\beta}{2} \sin \varphi}, \text{ si } \cot \varphi = \frac{b \sin \frac{\alpha+\beta}{2}}{a \sin \frac{\alpha}{2} \sin \frac{\beta}{2}}.$$

$$\text{Porro } a = \varrho (\cot \frac{\alpha}{2} + \cot \frac{\beta}{2}) = \frac{\varrho \sin \frac{\alpha+\beta}{2}}{\sin \frac{\alpha}{2} \sin \frac{\beta}{2}}; \text{ igitur}$$

$$\varrho = \frac{a \sin \frac{\alpha}{2} \sin \frac{\beta}{2}}{\sin \frac{\alpha+\beta}{2}}$$

$$J = (b + d) \varrho$$

$$= \frac{b^2 \sin^2 \frac{\alpha+\beta}{2} - ab (\sin^2 \frac{\alpha}{2} - \sin^2 \frac{\beta}{2})}{b \sin^2 \frac{\alpha+\beta}{2} - a^2 \sin^2 \frac{\alpha}{2}} \times \frac{a \sin \frac{\alpha}{2} \sin \frac{\beta}{2}}{\sin \frac{\alpha+\beta}{2}}$$

$$J = \frac{[b \sin \frac{\alpha+\beta}{2} - a \sin \frac{\alpha-\beta}{2}] \, ab \sin \frac{\alpha}{2} \sin \frac{\beta}{2}}{b \sin^2 \frac{\alpha+\beta}{2} - a^2 \sin^2 \frac{\alpha}{2}}$$

9) **Problema 3.** Datis 2 lateribus finitimis (a, b) et 2 angulis, qui ad unum laterum non datorum adjacent (α, δ), quaerantur latera non data (c, d), anguli non dati (β, γ), area (J), radius circuli (ϱ).

Solutio. Secundum §. 14. est $a \sin\frac{\alpha}{2} \sin\frac{\beta+\gamma}{2} = b \sin\frac{\gamma}{2} \sin\frac{\alpha+\beta}{2}$,

sive $a \sin\frac{\alpha}{2} \sin\frac{\alpha+\delta}{2} = b \sin\frac{\gamma}{2} \sin\frac{\gamma+\delta}{2} = \frac{1}{2} b \cos\frac{\delta}{2} - \frac{1}{2} b \cos(\gamma + \frac{\delta}{2})$.

$$\text{Igitur } \cos(\gamma + \frac{\delta}{2}) = \frac{b \cos\frac{\delta}{2} - 2a \sin\frac{\alpha}{2} \sin\frac{\alpha+\delta}{2}}{b} = \frac{\cos\frac{1}{2}\delta \cos 2\varphi}{\cos^2\varphi},$$

$$\text{si tang } \varphi = \sqrt{\frac{2a \sin\frac{\alpha}{2} \sin\frac{\alpha+\delta}{2}}{b \cos\frac{\delta}{2}}}.$$

Angulus γ, igitur etiam β 2 valores assumere potest, quamobrem 2 figurae quadrangulae oriuntur.

Ex §. 17. sequitur $ac \sin^2\frac{\alpha+\delta}{2} = c\,(a+c-b) \sin^2\frac{\delta}{2} + a\,(a+c-b) \sin^2\frac{\alpha}{2}$,

$$\text{igitur } c = \frac{2a \sin\frac{\alpha}{2} \cos\frac{\alpha+\delta}{2} + b \sin\frac{\delta}{2} \pm \sqrt{b^2 - (2a \sin\frac{\alpha}{2} \sin\frac{\alpha+\delta}{2} - b \cos\frac{\delta}{2})^2}}{2 \sin\frac{\delta}{2}}.$$

Quum haec forma ad calculum logarithmicum inutilis, et angulus β jam notus sit, ex §. 14. invenitur

$$c = \frac{b \sin\frac{\beta}{2} \sin\frac{\alpha+\beta}{2}}{\sin\frac{\delta}{2} \sin\frac{\alpha+\delta}{2}},$$

$$d = \frac{a \sin\frac{\beta}{2} \sin\frac{\alpha+\delta}{2}}{\sin\frac{\delta}{2} \sin\frac{\alpha+\beta}{2}}.$$

Si in aequatione $J = \frac{1}{2} ab \sin\beta + \frac{1}{2} cd \sin\delta$ secundum §. 16. $cd \sin^2\frac{\delta}{2} = ab \sin^2\frac{\beta}{2}$ substitueris, erit

$$J = \frac{ab \sin\frac{\beta}{2} \sin\frac{\beta+\delta}{2}}{\sin\frac{\delta}{2}},$$

ubi pro β valorem supra inventum substitui necesse est.

Idem fiat in aequatione

$$\varrho = \frac{a \sin \frac{\alpha}{2} \sin \frac{\beta}{2}}{\sin \frac{\alpha+\beta}{2}};$$

nam si ϱ quantitatibus datis definitur, est

$$\varrho = \frac{(2a-b) \sin \frac{\alpha}{2} \sin \frac{\delta}{2} + 2a \cos \frac{\alpha}{2} \cos \frac{\alpha+\delta}{2} \pm \sqrt{b^2 - (2a \sin \frac{\alpha}{2} \sin \frac{\alpha+\delta}{2} - b \cos^2 \frac{\delta}{2})^2}}{2 \sin \frac{\alpha+\delta}{2}}$$

10) Problema 4. Datis 2 lateribus finitimis (a, b) et 2 angulis sibi oppositis, quorum neuter est inclusus (α, γ), quaerantur latera non data (c, d), anguli non dati (β, δ), area (J), radius circuli (ϱ).

Solutio. Ex aequatione §. 14. $a \sin \frac{\alpha}{2} \sin \frac{\beta+\gamma}{2} = b \sin \frac{\gamma}{2} \sin \frac{\alpha+\beta}{2}$ sequitur:

$$\cot \frac{\beta}{2} = \frac{b \cot \frac{\alpha}{2} - a \cot \frac{\gamma}{2}}{a-b} = \frac{b \cot \frac{\alpha}{2} \cos 2\varphi}{(a-b) \cos^2 \varphi}, \text{ si } \operatorname{tang} \varphi = \sqrt{\frac{a \cot \frac{\gamma}{2}}{b \cot \frac{\alpha}{2}}}.$$

Quum sit $d - c = a - b$ et secundum §. 16. $\frac{c}{d} = \frac{a \sin^2 \frac{\alpha}{2}}{b \sin^2 \frac{\gamma}{2}}$, invenitur

$$c = \frac{a (a-b) \sin^2 \frac{\alpha}{2}}{b \sin^2 \frac{\gamma}{2} - a \sin^2 \frac{\alpha}{2}} = \frac{(a-b) \sin^2 \varphi}{\cos 2\varphi} \text{ et}$$

$$d = \frac{b (a-b) \sin^2 \frac{\gamma}{2}}{b \sin^2 \frac{\gamma}{2} - a \sin^2 \frac{\alpha}{2}} = \frac{(a-b) \cos^2 \varphi}{\cos 2\varphi}, \text{ si } \cot \varphi = \frac{\sin \frac{\gamma}{2}}{\sin \frac{\alpha}{2}} \sqrt{\frac{b}{a}}.$$

Si in aequatione $J = \frac{1}{2} ad \sin \alpha + \frac{1}{2} bc \sin \gamma$ valores laterum c et d substitueris, erit

$$J = \frac{ab (a-b) \sin \frac{\alpha}{2} \sin \frac{\gamma}{2} \sin \frac{\alpha+\gamma}{2}}{b \sin^2 \frac{\gamma}{2} - a \sin^2 \frac{\alpha}{2}} = \frac{a (a-b) \sin \frac{\alpha}{2} \sin \frac{\alpha+\gamma}{2} \cos^2 \varphi}{\sin \frac{\gamma}{2} \cos 2\varphi}$$

$$\varrho = \frac{J}{a+c}; \text{ est autem } a+c = \frac{ab\left(\sin^2\frac{\gamma}{2} - \sin^2\frac{\alpha}{2}\right)}{b\sin^2\frac{\gamma}{2} - a\sin^2\frac{\alpha}{2}} = \frac{ab\sin\frac{\gamma+\alpha}{2}\sin\frac{\gamma-\alpha}{2}}{b\sin^2\frac{\gamma}{2} - a\sin^2\frac{\alpha}{2}};$$

$$\text{ergo } \varrho = \frac{(a-b)\sin\frac{\alpha}{2}\sin\frac{\gamma}{2}}{\sin\frac{\gamma-\alpha}{2}}$$

11) Problema 5. Datis 2 lateribus finitimis (a, b) et 2 angulis sibi oppositis, inter quos angulus est inclusus (β, δ), quaerantur latera non data (c, d), anguli non dati (α, γ), area (J), radius circuli (ϱ).

Solutio. Quum sit $c - d = b - a$ et secundum §. 16.

$$cd = \frac{ab\sin^2\frac{\beta}{2}}{\sin^2\frac{\delta}{2}}, \quad c+d = \sqrt{(b-a)^2 + \frac{4ab\sin^2\frac{\beta}{2}}{\sin^2\frac{\delta}{2}}} = (b-a)\sec\varphi,$$

$$\text{si tang } \varphi = \frac{2\sin\frac{\beta}{2}\sqrt{ab}}{(b-a)\sin\frac{\delta}{2}}. \text{ Inde sequitur}$$

$$c = \frac{(b-a)\cos^2\frac{\varphi}{2}}{\cos\varphi}, \text{ et } d = \frac{(b-a)\sin^2\frac{\varphi}{2}}{\cos\varphi}.$$

Quum aequatio, qua anguli α et γ definiuntur, ad calculum logarithmicum inutilis sit, quaerantur anguli $\angle(ae)$ et $\angle(be)$ aequationibus $\text{tang}\,\frac{\angle(ae)-\angle(be)}{2} = \frac{b+a}{b-a}\cot\frac{\beta}{2}$ et $\angle(ae)+\angle(be) = 180^\circ - \beta$. Quum porro $c+d = \frac{b-a}{\cos\varphi}$, igitur $\frac{c+d}{c-d} = \frac{\frac{b-a}{\cos\varphi}}{b-a} = \frac{1}{\cos\varphi}$, quaerantur etiam anguli $\angle(ce)$ et $\angle(de)$ aequationibus $\text{tang}\,\frac{\angle(de)-\angle(ce)}{2} = \frac{\cot\frac{\delta}{2}}{\cos\varphi}$ et $\angle(de)+\angle(ce) = 180^\circ - \delta$. Si tum et β et $\delta < 180^\circ$, $\alpha = \angle(ae)+\angle(de)$ et $\gamma = \angle(be)+\angle(ce)$. Si $\beta > 180^\circ$, $\alpha = \angle(de)-\angle(ae)$, $\gamma = \angle(ce)-\angle(be)$. Si $\delta > 180^\circ$, $\alpha = \angle(ae)-\angle(de)$ et $\gamma = \angle(be)-\angle(ce)$.

Sicut in problemate 3. invenitur.

$$J = \frac{ab \sin\frac{\beta}{2} \sin\frac{\beta+\delta}{2}}{\sin\frac{\delta}{2}}.$$

Denique $\varrho = \frac{J}{a+c} = \frac{ab \sin\frac{\beta}{2} \sin\frac{\beta+\delta}{2}}{(a+c)\sin\frac{\delta}{2}}$. Est autem

$$a + c = \frac{a+b}{2} + \frac{1}{2}\sqrt{(b-a)^2 + \frac{4ab \sin^2\frac{\beta}{2}}{\sin^2\frac{\delta}{2}}}.$$ Igitur

$$\varrho = \frac{2ab \sin\frac{\beta}{2} \sin\frac{\beta+\delta}{2}}{(a+b)\sin\frac{\delta}{2} + \sqrt{(b-a)^2 \sin^2\frac{\delta}{2} + 4ab \sin^2\frac{\beta}{2}}}$$

$$= \frac{(a+b)\sin\frac{\beta}{2}\sin\frac{\delta}{2} - (b-a)\sin\frac{\beta}{2}\sin\frac{\delta}{2}\sqrt{1 + \frac{4ab\sin^2\frac{\beta}{2}}{(b-a)^2\sin^2\frac{\delta}{2}}}}{2\sin\frac{\delta-\beta}{2}}$$

$$\text{sive } \varrho = \frac{\left[(b+a) - \frac{b-a}{\sin\varphi}\right]\sin\frac{\beta}{2}\sin\frac{\delta}{2}}{2\sin\frac{\delta-\beta}{2}}$$

12) Problema 6. Datis 2 lateribus oppositis (a, c) et 2 angulis, qui ad unum eorum adjacent (α, β), quaerantur latera non data (b, d), anguli non dati (γ, δ), area (J), radius circuli (ϱ).

Solutio. Secundum §. 15. est $a \sin\frac{\alpha}{2} \sin\frac{\beta}{2} = c \sin\frac{\gamma}{2} \sin\frac{\alpha+\beta+\gamma}{2}$.

Igitur $$\frac{a \sin\frac{\alpha}{2}\sin\frac{\beta}{2}}{c} = \sin^2\frac{\gamma}{2}\cos\frac{\alpha+\beta}{2} + \sin\frac{\gamma}{2}\cos\frac{\gamma}{2}\sin\frac{\alpha+\beta}{2}$$

$$= \frac{\cos\frac{\alpha+\beta}{2} - \cos\gamma\cos\frac{\alpha+\beta}{2} + \sin\gamma\sin\frac{\alpha+\beta}{2}}{2} = \frac{\cos\frac{\alpha+\beta}{2} - \cos\frac{\alpha+\beta+2\gamma}{2}}{2},$$

unde sequitur:

$$\cos\frac{\alpha+\beta+2\gamma}{2} = \frac{c\cos\frac{\alpha+\beta}{2} - 2a\sin\frac{\alpha}{2}\sin\frac{\beta}{2}}{c} = \frac{c\cos\frac{\alpha}{2}\cos\frac{\beta}{2} - (2a+c)\sin\frac{\alpha}{\beta}\sin\frac{\beta}{2}}{c}$$

$$= \frac{\cos\frac{\alpha}{2}\sin\left(\varphi - \frac{\beta}{2}\right)}{\sin\varphi}, \quad \text{si } \cot\varphi = \frac{2a+c}{c}\tang\frac{\alpha}{2}.$$

Quum haec aequatio 2 valores anguli γ praebeat, 2 figurae quandrangulae problemati respondent.

Ex aequatione §. 17. $bd\sin^2\frac{\alpha+\beta}{2} = ad\sin^2\frac{\alpha}{2} + ab\sin^2\frac{\beta}{2}$,

sive $b\,(a-b+c)\sin^2\frac{\alpha+\beta}{2} = a\,(a-b+c)\sin^2\frac{\alpha}{2} + ab\sin^2\frac{\beta}{2}$ deducitur

$$b = \frac{(a+c)\sin\frac{\alpha+\beta}{2} + a\sin\frac{\alpha-\beta}{2} \pm \sqrt{\left[(a+c)\sin\frac{\alpha+\beta}{2} + a\sin\frac{\alpha-\beta}{2}\right]^2 - 4a\,(a+c)\sin^2\frac{\alpha}{2}}}{2\sin\frac{\alpha+\beta}{2}}$$

Quum autem haec forma ad calculum logarithmicum inutilis et anguli jam inventi sint, latera b et d quaerantur aequationibus

$$b = \frac{a\sin\frac{\alpha}{2}\sin\frac{\beta+\gamma}{2}}{\sin\frac{\gamma}{2}\sin\frac{\alpha+\beta}{2}}, \qquad d = \frac{a\sin\frac{\beta}{2}\sin\frac{\alpha+\delta}{2}}{\sin\frac{\delta}{2}\sin\frac{\alpha+\beta}{2}}.$$

$$\varrho = \frac{a\sin\frac{\alpha}{2}\sin\frac{\beta}{2}}{\sin\frac{\alpha+\beta}{2}}$$

$$J = (a+c)\,\varrho = \frac{a\,(a+c)\sin\frac{\alpha}{2}\sin\frac{\beta}{2}}{\sin\frac{\alpha+\beta}{2}}$$

13) Problema 7. Datis 2 lateribus oppositis (a, c) et 2 angulis, qui ad latus non datum adjacent (α, δ), quaerantur latera non data (b, d), anguli non dati (β, γ), area (J), radius circuli (ϱ).

Solutio. Secundum §. 17. est $ac\sin^2\frac{\alpha+\delta}{2} = cd\sin^2\frac{\delta}{2} + ad\sin^2\frac{\alpha}{2}$,

$$\text{igitur } d = \frac{ac\sin^2\frac{\alpha+\delta}{2}}{a\sin^2\frac{\alpha}{2} + c\sin^2\frac{\delta}{2}} = \frac{c\sin^2\frac{\alpha+\delta}{2}\cos^2\varphi}{\sin^2\frac{\alpha}{2}}, \quad \text{si } \tang\varphi = \frac{\sin\frac{\delta}{2}}{\sin\frac{\alpha}{2}}\sqrt{\frac{c}{a}}.$$

$b = a + c - d.$

Ex aequatione §. 15. $a \sin \frac{\alpha}{2} \sin \frac{\alpha+\gamma+\delta}{2} = c \sin \frac{\gamma}{2} \sin \frac{\delta}{2}$ sequitur

$$\cot \frac{\gamma}{2} = \frac{c \sin \frac{\delta}{2} - a \sin \frac{\alpha}{2} \cos \frac{\alpha+\delta}{2}}{a \sin \frac{\alpha}{2} \sin \frac{\alpha+\delta}{2}} = \frac{\sin\left(\frac{\alpha+\delta}{2} - \varphi\right)}{\sin \frac{\alpha+\delta}{2} \sin \varphi}, \text{ si } \cot \varphi = \frac{c \sin \frac{\delta}{2}}{a \sin \frac{\alpha}{2} \sin \frac{\alpha+\delta}{2}}$$

Eadem ratione invenitur

$$\cot \frac{\beta}{2} = \frac{a \sin \frac{\alpha}{2} - c \sin \frac{\delta}{2} \cos \frac{\alpha+\delta}{2}}{c \sin \frac{\delta}{2} \sin \frac{\alpha+\delta}{2}} = \frac{\sin\left(\frac{\alpha+\delta}{2} - \psi\right)}{\sin \frac{\alpha+\delta}{2} \sin \psi}, \text{ si } \cot \psi = \frac{a \sin \frac{\alpha}{2}}{c \sin \frac{\delta}{2} \sin \frac{\alpha+\delta}{2}}$$

Si in aequatione $\varrho = \frac{d \sin \frac{\alpha}{2} \sin \frac{\delta}{2}}{\sin \frac{\alpha+\delta}{2}}$ valorem lateris d supra inventum posueris, erit

$$\varrho = \frac{ac \sin \frac{\alpha}{2} \sin \frac{\delta}{2} \sin \frac{\alpha+\delta}{2}}{a \sin^2 \frac{\alpha}{2} + c \sin^2 \frac{\delta}{2}} = \frac{c \sin \frac{\delta}{2} \sin \frac{\alpha+\delta}{2} \cos^2 \varphi}{\sin \frac{\alpha}{2}}$$

$$J = (a+c)\,\varrho = \frac{ac\,(a+c) \sin \frac{\alpha}{2} \sin \frac{\delta}{2} \sin \frac{\alpha+\delta}{2}}{a \sin^2 \frac{\alpha}{2} + c \sin^2 \frac{\delta}{2}} = \frac{c\,(a+c) \sin \frac{\delta}{2} \sin \frac{\alpha+\delta}{2} \cos^2 \varphi}{\sin \frac{\alpha}{2}}.$$

14. Problema 8. Datis 2 lateribus oppositis (a, c) et 2 angulis oppositis (α, γ), quaerantur latera non data (b, d), anguli non dati (β, δ), area (J), radius circuli (ϱ).

Solutio. Ex aequatione §. 16. $ad \sin^2 \frac{\alpha}{2} = c\,(c+a-d) \sin^2 \frac{\gamma}{2}$ sequitur

$$d = \frac{c\,(a+c) \sin^2 \frac{\gamma}{2}}{a \sin^2 \frac{\alpha}{2} + c \sin^2 \frac{\gamma}{2}} = (a+c) \cos^2 \varphi, \text{ si tang } \varphi = \frac{\sin \frac{\alpha}{2}}{\sin \frac{\gamma}{2}} \sqrt{\frac{a}{c}}$$

$$b = a + c - d = \frac{a\,(a+c) \sin^2 \frac{\alpha}{2}}{a \sin^2 \frac{\alpha}{2} + c \sin^2 \frac{\gamma}{2}} = (a+c) \sin^2 \varphi.$$

Ex aequatione §. 15. $a \sin\frac{\alpha}{2} \sin\frac{\beta}{2} = c \sin\frac{\gamma}{2} \sin\frac{\alpha+\beta+\gamma}{2}$ deducitur

$$\cot\frac{\beta}{2} = \frac{a \sin\frac{\alpha}{2} - c \sin\frac{\gamma}{2} \cos\frac{\alpha+\gamma}{2}}{c \sin\frac{\gamma}{2} \sin\frac{\alpha+\gamma}{2}} = \frac{\sin\left(\frac{\alpha+\gamma}{2} - \psi\right)}{\sin\frac{\alpha+\gamma}{2} \sin\psi}, \text{ si } \cot\psi = \frac{a \sin\frac{\alpha}{2}}{c \sin\frac{\gamma}{2} \sin\frac{\alpha+\gamma}{2}}$$

Eodem modo invenitur

$$\cot\frac{\gamma}{2} = \frac{c \sin\frac{\gamma}{2} - a \sin\frac{\alpha}{2} \sin\frac{\alpha+\gamma}{2}}{a \sin\frac{\alpha}{2} \sin\frac{\alpha+\gamma}{2}} = \frac{\sin\left(\frac{\alpha+\gamma}{2} - \omega\right)}{\sin\frac{\alpha+\gamma}{2} \sin\omega}, \text{ si } \cot\omega = \frac{c \sin\frac{\gamma}{2}}{a \sin\frac{\alpha}{2} \sin\frac{\alpha+\gamma}{2}}$$

Valoribus modo repertis loco laterum b et d positis aequatio $J = \frac{1}{2} ad \sin\alpha + \frac{1}{2} bc \sin\gamma$ mutatur in

$$J = \frac{ac\,(a+c) \sin\frac{\alpha}{2} \sin\frac{\gamma}{2} \sin\frac{\alpha+\gamma}{2}}{a \sin^2\frac{\alpha}{2} + c \sin^2\frac{\gamma}{2}} = \frac{a\,(a+c) \sin\frac{\alpha}{2} \sin\frac{\alpha+\gamma}{2} \cos^2\varphi}{\sin\frac{\gamma}{2}}$$

$$\varrho = \frac{J}{a+c} = \frac{ac \sin\frac{\alpha}{2} \sin\frac{\gamma}{2} \sin\frac{\alpha+\gamma}{2}}{a \sin^2\frac{\alpha}{2} + c \sin^2\frac{\gamma}{2}} = \frac{a \sin\frac{\alpha}{2} \sin\frac{\alpha+\gamma}{2} \cos^2\varphi}{\sin\frac{\gamma}{2}}$$

15) **Problema 9.** Datis 1 latere *(a)* et 3 angulis *(α, β, γ)*, quaerantur latera non data *(b, c, d)*, area *(J)*, radius circuli *(ϱ)*.

Solutio. Ex §. 14. sequitur

$$b = \frac{a \sin\frac{\alpha}{2} \sin\frac{\beta+\gamma}{2}}{\sin\frac{\gamma}{2} \sin\frac{\alpha+\beta}{2}}, \qquad d = \frac{a \sin\frac{\beta}{2} \sin\frac{\beta+\gamma}{2}}{\sin\frac{\alpha+\beta}{2} \sin\frac{\alpha+\beta+\gamma}{2}}$$

et ex §. 15. $$c = \frac{a \sin\frac{\alpha}{2} \sin\frac{\beta}{2}}{\sin\frac{\gamma}{2} \sin\frac{\alpha+\beta+\gamma}{2}}$$

$$\varrho = \frac{a \sin\frac{\alpha}{2} \sin\frac{\beta}{2}}{\sin\frac{\alpha+\beta}{2}}$$

In problemate 5. inventum est $J = \dfrac{ab \sin\frac{\beta}{2} \sin\frac{\beta+\delta}{2}}{\sin\frac{\delta}{2}}$; igitur

$$J = \frac{a^2 \sin\frac{\alpha}{2} \sin\frac{\beta}{2} \sin\frac{\alpha+\gamma}{2} \sin\frac{\beta+\gamma}{2}}{\sin\frac{\gamma}{2} \sin\frac{\alpha+\beta}{2} \sin\frac{\alpha+\beta+\gamma}{2}}.$$

Nota. Si in aequatione $J = (a + c)\,\varrho$ loco lateris c posueris

$\dfrac{a \sin\frac{\alpha}{2} \sin\frac{\beta}{2}}{\sin\frac{\gamma}{2} \sin\frac{\alpha+\beta+\gamma}{2}}$, erit $J = \dfrac{a^2 \sin\frac{\alpha}{2} \sin\frac{\beta}{2}}{\sin\frac{\gamma}{2} \sin\frac{\alpha+\beta}{2} \sin\frac{\alpha+\beta+\gamma}{2}} \left[\sin\frac{\alpha}{2} \sin\frac{\beta}{2} + \sin\frac{\gamma}{2} \sin\frac{\alpha+\beta+\gamma}{2}\right]$.

Est igitur $\sin\frac{\alpha}{2} \sin\frac{\beta}{2} + \sin\frac{\gamma}{2} \sin\frac{\alpha+\beta+\gamma}{2} = \sin\frac{\alpha+\gamma}{2} \sin\frac{\beta+\gamma}{2}$, quod idem nonnullis mutationibus facilibus probatur.

Eodem modo invenitur $\cos\frac{\alpha}{2} \cos\frac{\beta}{2} + \cos\frac{\gamma}{2} \cos\frac{\alpha+\beta+\gamma}{2} = \sin\frac{\alpha+\gamma}{2} \sin\frac{\beta+\gamma}{2}$.

Est igitur in quaque figura quadrangula primi ut secundi generis

I. $\sin\frac{\alpha}{2} \sin\frac{\beta}{2} + \sin\frac{\gamma}{2} \sin\frac{\delta}{2} = \cos\frac{\alpha}{2} \cos\frac{\beta}{2} + \cos\frac{\gamma}{2} \cos\frac{\delta}{2} = \sin\frac{\alpha+\gamma}{2} \sin\frac{\beta+\gamma}{2}$.

II. $\cos\frac{\alpha}{2} \cos\frac{\beta}{2} - \sin\frac{\gamma}{2} \sin\frac{\delta}{2} = \cos\frac{\alpha+\gamma}{2} \cos\frac{\beta+\gamma}{2}$.

III. $\sin\frac{\alpha}{2} \cos\frac{\beta}{2} - \sin\frac{\gamma}{2} \cos\frac{\delta}{2} = \sin\frac{\alpha+\gamma}{2} \cos\frac{\beta+\gamma}{2}$.

IV. $\sin\alpha \sin\beta - \sin\gamma \sin\delta = \cos\alpha \cos\beta - \cos\gamma \cos\delta = \sin(\alpha+\gamma) \sin(\beta+\gamma)$.

V. $\cos\alpha \cos\beta + \sin\gamma \sin\delta = \sin\alpha \sin\beta + \cos\gamma \cos\delta = \cos(\alpha+\gamma) \cos(\beta+\gamma)$.

VI. $\sin\alpha \cos\beta + \sin\gamma \cos\delta = \sin(\alpha+\gamma) \cos(\beta+\gamma)$.

Ex I. et III. sequitur

VII. $$\frac{\sin\frac{\alpha}{2} \sin\frac{\beta}{2} + \sin\frac{\gamma}{2} \sin\frac{\delta}{2}}{\sin\frac{\alpha}{2} \cos\frac{\beta}{2} - \sin\frac{\gamma}{2} \cos\frac{\delta}{2}} = \frac{\cos\frac{\alpha}{2} \cos\frac{\beta}{2} + \cos\frac{\gamma}{2} \cos\frac{\delta}{2}}{\sin\frac{\alpha}{2} \cos\frac{\beta}{2} - \sin\frac{\gamma}{2} \cos\frac{\delta}{2}} = \operatorname{tang}\frac{\beta+\gamma}{2}$$

Ex II. et III.

VIII. $$\frac{\cos\frac{\alpha}{2} \cos\frac{\beta}{2} - \sin\frac{\gamma}{2} \sin\frac{\delta}{2}}{\sin\frac{\alpha}{2} \cos\frac{\beta}{2} - \sin\frac{\gamma}{2} \cos\frac{\delta}{2}} = \cot\frac{\alpha+\gamma}{2}$$

Ex IV. et VI.

$$\text{IX.}\quad \frac{\sin\alpha\sin\beta-\sin\gamma\sin\delta}{\sin\alpha\cos\beta+\sin\gamma\cos\delta}=\frac{\cos\alpha\cos\beta-\cos\gamma\cos\delta}{\sin\alpha\cos\beta+\sin\gamma\cos\delta}=\operatorname{tang}(\beta+\gamma).$$

Ex V. et VI.

$$\text{X.}\quad \frac{\cos\alpha\cos\beta+\sin\gamma\sin\delta}{\sin\alpha\cos\beta+\sin\gamma\cos\delta}=\frac{\sin\alpha\sin\beta+\cos\gamma\cos\delta}{\sin\alpha\cos\beta+\sin\gamma\cos\delta}=\cot(\alpha+\gamma).$$

Ex I. et IV.

$$\text{XI.}\quad \frac{\sin\alpha\sin\beta-\sin\gamma\sin\delta}{\sin\frac{\alpha}{2}\sin\frac{\beta}{2}+\sin\frac{\gamma}{2}\sin\frac{\delta}{2}}=\frac{\cos\alpha\cos\beta-\cos\gamma\cos\delta}{\cos\frac{\alpha}{2}\cos\frac{\beta}{2}+\cos\frac{\gamma}{2}\cos\frac{\delta}{2}}=4\cos\frac{\alpha+\gamma}{2}\cos\frac{\beta+\gamma}{2}.$$

Ex II. et IV.

$$\text{XII.}\quad \frac{\sin\alpha\sin\beta-\sin\gamma\sin\delta}{\cos\frac{\alpha}{2}\cos\frac{\beta}{2}-\sin\frac{\gamma}{2}\sin\frac{\delta}{2}}=\frac{\cos\alpha\cos\beta-\cos\gamma\cos\delta}{\cos\frac{\alpha}{2}\cos\frac{\beta}{2}-\sin\frac{\gamma}{2}\sin\frac{\delta}{2}}=4\sin\frac{\alpha+\gamma}{2}\sin\frac{\beta+\gamma}{2}.$$

etc.

Quum, ut notissimum, sit

$$\begin{aligned}8\sin u\sin x\sin y\sin z=&\cos(u+x+y+z)-\cos(u+x+y-z)-\cos(u+x-y+z)\\&-\cos(u-x+y+z)-\cos(-u+x+y+z)+\cos(u+x-y-z)\\&+\cos(u-x-y+z)+\cos(u-x+y-z)\end{aligned}$$

et

$$\begin{aligned}8\cos u\cos x\cos y\cos z=&\cos(u+x+y+z)+\cos(u+x+y-z)+\cos(u+x-y+z)\\&+\cos(u-x+y+z)+\cos(-u+x+y+z)+\cos(u+x-y-z)\\&+\cos(u-x-y+z)+\cos(u-x+y-z),\end{aligned}$$

si pro u, x, y z anguli figurae quadrangulae ponuntur, est

$$8\sin\alpha\sin\beta\sin\gamma\sin\delta=1-\cos2\alpha-\cos2\beta-\cos2\gamma-\cos2\delta+\cos2(\alpha+\beta)+\cos2(\alpha+\gamma)+\cos2(\alpha+\delta)$$

et

$$8\cos\alpha\cos\beta\cos\gamma\cos\delta=1+\cos2\alpha+\cos2\beta+\cos2\gamma+\cos2\delta+\cos2(\alpha+\beta)+\cos2(\alpha+\gamma)+\cos2(\alpha+\delta).$$

Igitur

$$4\sin\alpha\sin\beta\sin\gamma\sin\delta+4\cos\alpha\cos\beta\cos\gamma\cos\delta=1+\cos2(\alpha+\beta)+\cos2(\alpha+\gamma)+\cos2(\alpha+\delta)$$

sive

$$\text{XIII.}\quad \sin\alpha\sin\beta\sin\gamma\sin\delta+\cos\alpha\cos\beta\cos\gamma\cos\delta=\frac{\cos^2(\alpha+\beta)+\cos^2(\alpha+\gamma)+\cos^2(\alpha+\delta)-1}{2}$$

$$=\frac{2-\sin^2(\alpha+\beta)-\sin^2(\alpha+\gamma)-\sin^2(\alpha+\delta)}{2}.$$

Porro

$$4\cos\alpha\cos\beta\cos\gamma\cos\delta-4\sin\alpha\sin\beta\sin\gamma\sin\delta=\cos2\alpha+\cos2\beta+\cos2\gamma+\cos2\delta,$$

XIV. $1 + \sin\alpha \sin\beta \sin\gamma \sin\delta - \cos\alpha \cos\beta \cos\gamma \cos\delta = \frac{\sin^2\alpha + \sin^2\beta + \sin^2\gamma + \sin^2\delta}{2}$

et

XV. $1 - \sin\alpha \sin\beta \sin\gamma \sin\delta + \cos\alpha \cos\beta \cos\gamma \cos\delta = \frac{\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta}{2}$

Si autem pro u, x, y, z posueris $\frac{\alpha}{4}, \frac{\beta}{4}, \frac{\gamma}{4}, \frac{\delta}{4}$, evadent aequationes

XVI. $\cos\frac{\alpha}{4}\cos\frac{\beta}{4}\cos\frac{\gamma}{4}\cos\frac{\delta}{4} + \sin\frac{\alpha}{4}\sin\frac{\beta}{4}\sin\frac{\gamma}{4}\sin\frac{\delta}{4} = \frac{\sin\frac{\alpha+\beta}{2} + \sin\frac{\alpha+\gamma}{2} + \sin\frac{\alpha+\delta}{2}}{4}$

et

XVII. $\cos\frac{\alpha}{4}\cos\frac{\beta}{4}\cos\frac{\gamma}{4}\cos\frac{\delta}{4} - \sin\frac{\alpha}{4}\sin\frac{\beta}{4}\sin\frac{\gamma}{4}\sin\frac{\delta}{4} = \frac{\sin\frac{\alpha}{2} + \sin\frac{\beta}{2} + \sin\frac{\gamma}{2} + \sin\frac{\delta}{2}}{4}$

etc.

Quum sit $\operatorname{tang} u + \operatorname{tang} x + \operatorname{tang} y = \frac{\sin(u+x)\cos y + \cos u \cos x \sin y}{\cos u \cos x \cos y}$

$$= \frac{\sin(u+x)\cos y + [\cos(u+x) + \sin u \sin x]\sin y}{\cos u \cos x \cos y}$$

i. e. $\operatorname{tang} u + \operatorname{tang} x + \operatorname{tang} y = \frac{\sin(u+x+y)}{\cos u \cos x \cos y} + \operatorname{tang} u \operatorname{tang} x \operatorname{tang} y$, est quoque

$$\operatorname{tang} u + \operatorname{tang} x + \operatorname{tang} y + \operatorname{tang} z = \frac{\sin(u+x+y)\cos z + \sin u \sin x \sin y \cos z + \cos u \cos x \cos y \sin z}{\cos u \cos x \cos y \cos z}$$

Sed $\cos(u+x+y) = \cos u \cos x \cos y - \sin u \sin x \cos y - \sin u \cos x \sin y - \cos u \sin x \sin y$

Ergo $\operatorname{tang} u + \operatorname{tang} x + \operatorname{tang} y + \operatorname{tang} z$

$$= \frac{\sin(u+x+y)\cos z + \cos(u+x+y)\sin z + \sin u \sin x \sin y \cos z}{\cos u \cos x \cos y \cos z}$$

$$+ \frac{\sin u \sin x \cos y \sin z + \sin u \cos x \sin y \sin z + \cos u \sin x \sin y \sin z}{\cos u \cos x \cos y \cos z}$$

i. e. $\operatorname{tang} u + \operatorname{tang} x + \operatorname{tang} y + \operatorname{tang} z = \frac{\sin(u+x+y+z)}{\cos u \cos x \cos y \cos z} + \operatorname{tang} u \operatorname{tang} x \operatorname{tang} y$

$+ \operatorname{tang} u \operatorname{tang} x \operatorname{tang} z + \operatorname{tang} u \operatorname{tang} y \operatorname{tang} z + \operatorname{tang} x \operatorname{tang} y \operatorname{tang} z.$

Si igitur pro u, x, y, z anguli figurae quadranguli ponuntur, oritur aequatio

XVIII. $\operatorname{tang}\alpha + \operatorname{tang}\beta + \operatorname{tang}\gamma + \operatorname{tang}\delta = \operatorname{tang}\alpha \operatorname{tang}\beta \operatorname{tang}\gamma + \operatorname{tang}\alpha \operatorname{tang}\beta \operatorname{tang}\delta$

$+ \operatorname{tang}\alpha \operatorname{tang}\gamma \operatorname{tang}\delta + \operatorname{tang}\beta \operatorname{tang}\gamma \operatorname{tang}\delta$

et si ponitur $u = \frac{\alpha}{2}, x = \frac{\beta}{2}, y = \frac{\gamma}{2}, z = \frac{\delta}{2}$,

$$\text{XIX.}\quad \operatorname{tang}\frac{\alpha}{2}+\operatorname{tang}\frac{\beta}{2}+\operatorname{tang}\frac{\gamma}{2}+\operatorname{tang}\frac{\delta}{2}=\operatorname{tang}\frac{\alpha}{2}\operatorname{tang}\frac{\beta}{2}\operatorname{tang}\frac{\gamma}{2}+\operatorname{tang}\frac{\alpha}{2}\operatorname{tang}\frac{\beta}{2}\operatorname{tang}\frac{\delta}{2}$$
$$+\operatorname{tang}\frac{\alpha}{2}\operatorname{tang}\frac{\gamma}{2}\operatorname{tang}\frac{\delta}{2}+\operatorname{tang}\frac{\beta}{2}\operatorname{tang}\frac{\gamma}{2}\operatorname{tang}\frac{\delta}{2}$$

Eadem ratione invenitur

$$\cot u+\cot x+\cot y+\cot z=\cot u\cot x\cot y+\cot u\cot x\cot z+\cot u\cot y\cot z$$
$$+\cot x\cot y\cot z-\frac{\sin(u+x+y+z)}{\sin u\sin x\sin y\sin z}$$

Ergo

$$\text{XX.}\quad \cot\alpha+\cot\beta+\cot\gamma+\cot\delta=\cot\alpha\cot\beta\cot\gamma+\cot\alpha\cot\beta\cot\delta+\cot\alpha\cot\gamma\cot\delta+\cot\beta\cot\gamma\cot\delta$$

et

$$\text{XXI.}\quad \cot\frac{\alpha}{2}+\cot\frac{\beta}{2}+\cot\frac{\gamma}{2}+\cot\frac{\delta}{2}=\cot\frac{\alpha}{2}\cot\frac{\beta}{2}\cot\frac{\gamma}{2}+\cot\frac{\alpha}{2}\cot\frac{\beta}{2}\cot\frac{\delta}{2}$$
$$+\cot\frac{\alpha}{2}\cot\frac{\gamma}{2}\cot\frac{\delta}{2}+\cot\frac{\beta}{2}\cot\frac{\gamma}{2}\cot\frac{\delta}{2}.$$

Porro

$$\operatorname{tang}u+\operatorname{tang}x+\operatorname{tang}y-\cot z=\operatorname{tang}u\operatorname{tang}x\operatorname{tang}y-\operatorname{tang}u\operatorname{tang}x\cot z$$
$$-\operatorname{tang}u\operatorname{tang}y\cot z-\operatorname{tang}x\operatorname{tang}y\cot z-\frac{\cos(u+x+y+z)}{\cos u\cos x\cos y\cos z}$$

Ergo

$$\text{XXII.}\quad \operatorname{tang}\frac{\alpha}{4}+\operatorname{tang}\frac{\beta}{4}+\operatorname{tang}\frac{\gamma}{4}-\cot\frac{\delta}{4}=\operatorname{tang}\frac{\alpha}{4}\operatorname{tang}\frac{\beta}{4}\operatorname{tang}\frac{\gamma}{4}-\operatorname{tang}\frac{\alpha}{4}\operatorname{tang}\frac{\beta}{4}\cot\frac{\delta}{4}$$
$$-\operatorname{tang}\frac{\alpha}{4}\operatorname{tang}\frac{\gamma}{4}\cot\frac{\delta}{4}-\operatorname{tang}\frac{\beta}{4}\operatorname{tang}\frac{\gamma}{4}\cot\frac{\delta}{4}$$

$$\cot u+\cot x-\operatorname{tang}y-\operatorname{tang}z=\frac{\sin(u+x+y+z)}{\sin u\sin x\cos y\cos z}-\cot u\cot x\operatorname{tang}y$$
$$-\cot u\cot x\cot z+\cot u\operatorname{tang}y\operatorname{tang}z+\cot x\operatorname{tang}y\operatorname{tang}z.$$

Ergo

$$\text{XXIII.}\quad \cot\alpha+\cot\beta-\operatorname{tang}\gamma-\operatorname{tang}\delta=-\cot\alpha\cot\beta\operatorname{tang}\gamma-\cot\alpha\cot\beta\operatorname{tang}\delta$$
$$+\cot\alpha\operatorname{tang}\gamma\operatorname{tang}\delta+\cot\beta\operatorname{tang}\gamma\operatorname{tang}\delta$$

et

$$\text{XXIV.}\quad \cot\frac{\alpha}{2}+\cot\frac{\beta}{2}-\operatorname{tang}\frac{\gamma}{2}-\operatorname{tang}\frac{\delta}{2}=-\cot\frac{\alpha}{2}\cot\frac{\beta}{2}\operatorname{tang}\frac{\gamma}{2}-\cot\frac{\alpha}{2}\cot\frac{\beta}{2}\operatorname{tang}\frac{\delta}{2}$$
$$+\cot\frac{\alpha}{2}\operatorname{tang}\frac{\gamma}{2}\operatorname{tang}\frac{\delta}{2}+\cot\frac{\beta}{2}\operatorname{tang}\frac{\gamma}{2}\operatorname{tang}\frac{\delta}{2}$$

Figurae quadrángulae, in quibus differentiae laterum oppositorum aequales, omittantur, quia aequationes, quae in eas valent, eodem modo ex §§. 37—42 deducuntur.

b. Figurae quadrangulae, in quibus et summae et differentiae laterum oppositorum aequales sunt. $(a+c=b+d,\ a-c=d-b.)$

16) In has figuras quadrangulas 2 circuli perscribi possunt, qui omnia latera aut ipsa aut producta tangant.

Quum in iis $a=d$, $b=c$, $\beta=\delta$, 3 quantitatibus definiuntur. Data igitur esse possunt.

1) $a,\ b,\ \beta.$ 3) $a,\ \alpha,\ \gamma.$

2) $a,\ b,\ \alpha.$ 4) $a,\ \alpha.\ \beta.$

ϱ^1 denotet radium circuli, qui extra figuram situs est.

17) Problema 1. Datis 2 lateribus inaequalibus (a, b) et 1 angulorum aequalium (β), quaerantur anguli non dati (α, γ), area (J), radii circulorum (ϱ, ϱ^1).

Solutio. Ex aequatione §. 43. $b \sin \frac{\gamma}{2} = a \sin \frac{\gamma+2\beta}{2}$ sequitur

$$\cot \frac{\gamma}{2} = \frac{b-a\cos\beta}{a\sin\beta} = \frac{\sin(\beta-\varphi)}{\sin\beta\ \sin\varphi},\ \text{si}\ \cot\varphi = \frac{b}{a\sin\beta},$$

et ex aequatione $a \sin \frac{\alpha}{2} = b \sin \frac{\alpha+2\beta}{2}$

$$\cot \frac{\alpha}{2} = \frac{a-b\cos\beta}{b\sin\beta} = \frac{\sin(\beta-\psi)}{\sin\beta\sin\psi},\quad \text{si}\ \cot\psi = \frac{a}{b\sin\beta}.$$

Aut quaerantur anguli α, γ aequationibus

$$\operatorname{tang}\frac{\alpha-\gamma}{4} = \frac{b-a}{b+a}\cot\frac{\beta}{2}\quad \text{et}\quad \frac{\alpha+\gamma}{2} = 180^\circ - \beta.$$

$J = \frac{1}{2}ab \sin\beta + \frac{1}{2}cd \sin\gamma$. Sed in his figuris $ab \sin\beta = cd \sin\gamma$, ergo

$$J = ab \sin\beta,$$

$$\varrho = \frac{J}{a+c} = \frac{ab\sin\beta}{a+b}$$

Quum $J = \frac{1}{2}(a-b-c-d)\varrho^1 = (a-b)\varrho^1$,

$$\varrho^1 = \frac{J}{a-b} = \frac{ab\sin\beta}{a-b}.$$

18) Problema 2. Datis 2 lateribus inaequalibus (a, b) et 1 angulorum inaequalium (α), quaerantur anguli non dati (β, γ), area (J), radii circulorum (ϱ, ϱ^1).

Solutio. Ex §. 43. sequitur $\sin\frac{\gamma}{2} = \frac{a\sin\frac{\alpha}{2}}{b}$

$$\text{et}\ \sin\beta = \frac{a\cos\frac{\alpha}{2} \pm \sqrt{b^2 - a^2\sin^2\frac{\alpha}{2}}}{b}\sin\frac{\alpha}{2},\quad \text{aut}\ \beta = 180^\circ - \frac{\alpha+\gamma}{2}$$

$$J = ab \sin\beta = (a \cos\frac{\alpha}{2} \pm \sqrt{b^2 - a^2 \sin^2\frac{\alpha}{2}})\, a \sin\frac{\alpha}{2}$$

$$\varrho = \frac{ab \sin\beta}{a+b} = \frac{a \cos\frac{\alpha}{2} \pm \sqrt{b^2 - a^2 \sin^2\frac{\alpha}{2}}}{a+b}\, a \sin\frac{\alpha}{2}$$

$$\varrho^1 = \frac{ab \sin\beta}{a-b} = \frac{a \cos\frac{\alpha}{2} \pm \sqrt{b^2 - a^2 \sin^2\frac{\alpha}{2}}}{a-b}\, a \sin\frac{\alpha}{2}.$$

$b > a \sin\frac{\alpha}{2}$ esse oportet. Si $\alpha < 180^\circ$ et $a > b$, 1 figura quadrangula primi et 1 tertii generis problemati satisfacit, si autem $a < b$ 1 primi generis.

19) Problema 3. Datis 1 latere (a) et 2 angulis inaequalibus (α, γ), quaeratur latus non datum (b), angulus non datus (β), area (J), radii circulorum (ϱ, ϱ^1).

Solutio. $\beta = \delta = 180^\circ - \frac{\alpha+\gamma}{2}$ et secundum §. 43. $b = \frac{a \sin\frac{\alpha}{2}}{\sin\frac{\gamma}{2}}$

$$J = ab \sin\beta = \frac{a^2 \sin\frac{\alpha}{2} \sin\frac{\alpha+\gamma}{2}}{\sin\frac{\gamma}{2}}$$

$$\varrho = \frac{ab \sin\beta}{a+b} = \frac{a^2 \sin\frac{\alpha}{2} \sin\frac{\alpha+\gamma}{2}}{a\,(\sin\frac{\alpha}{2} + \sin\frac{\gamma}{2})} = \frac{a \sin\frac{\alpha}{2} \cos\frac{\alpha+\gamma}{4}}{\sin\frac{\alpha-\gamma}{4}}$$

$$\varrho^1 = \frac{ab \sin\beta}{a-b} = \frac{a \sin\frac{\alpha}{2} \sin\frac{\alpha+\gamma}{4}}{\sin\frac{\alpha-\gamma}{4}}.$$

20) Problema 4. Datis 1 latere (a), 1 angulorum inaequalium (α) et 1 aequalium (β), quaeratur latus non datum (b), angulus non datus (γ), area (J), radii circulorum (ϱ, ϱ^1).

Solutio. $\gamma = 360^\circ - (\alpha + 2\beta)$ et secundum §. 43. $b = \frac{a \sin\frac{\alpha}{2}}{\sin\frac{\alpha+\beta}{2}}$

$$J = ab \sin\beta = \frac{a^2 \sin\frac{\alpha}{2} \sin\beta}{\sin\frac{\alpha+2\beta}{2}} \qquad \varrho = \frac{ab \sin\beta}{a+b} = \frac{a \sin\frac{\alpha}{2} \sin\frac{\beta}{2}}{\sin\frac{\alpha+\beta}{2}} \qquad \varrho^1 = \frac{ab \sin\beta}{a-b} = \frac{a \sin\frac{\alpha}{2} \cos\frac{\beta}{2}}{\cos\frac{\alpha+\beta}{2}}$$

C. Figurae quadrangulae, circum quas et in quas circulus perscribi potest.

21. Si earum tantum figurarum rationem habemus, in quibus summae laterum oppositorum aequales sint, propter 2 aequationes $\alpha + \gamma = \beta + \delta$, $a + c = b + d$ primi generis esse debent et 3 quantitatibus definiuntur, inter quas summum 2 anguli iique finitimi. Data igitur esse possunt

1) a, b, c. 3) a, b, γ. 5) a, α, δ.
2) a, b, β. 4) a, c, α.

22) Problema 1. Datis 3 lateribus (a, b, c) quaerantur latus quartum (d), anguli $(\alpha, \beta, \gamma, \delta)$, area (J), radii circulorum (r, ϱ).

Solutio. $d = a + c - b$. Ex §. 27. sequitur

$$\operatorname{tang} \frac{\beta}{2} = \sqrt{\frac{c\,(a+c-b)}{ab}} = \cot \frac{\delta}{2}$$

$$\operatorname{tang} \frac{\gamma}{2} = \sqrt{\frac{a\,(a+c-b)}{bc}} = \cot \frac{\alpha}{2}$$

Nro. 2) inventum est $J = {}^1/_4 \sqrt{(a+b+c-d)\,(a+b-c+d)\,(a-b+c+d)\,(-a+b+c+d)}$

Ergo $J = \sqrt{abcd} = \sqrt{abc\,(a-b+c)}$

$$\varrho = \frac{J}{a+c} = \frac{1}{a+c}\sqrt{abc\,(a-b+c)}$$

$$r = \frac{1}{4J}\sqrt{(ab+cd)\,(ac+bd)\,(bc+ad)}$$

$$= \sqrt{\frac{[a\,(b+c) - c\,(b-c)]\,[a\,(b+c) - b\,(b-c)]\,[a\,(a+c) - b\,(a-c)]}{abc\,(a-b+c)}}$$

23) Problema 2. Datis 2 lateribus finitimis (a, b) et angulo iis incluso (β), quaerantur latera non data (c, d), anguli non dati (α, γ), area (J), radii circulorum (r, ϱ).

Solutio. Ex aequatione §. 27. $ab \operatorname{tang}^2 \frac{\beta}{2} = c\,(a + c - b)$ derivatur

$$c = \frac{b - a + \sqrt{(b-a)^2 + 4ab \operatorname{tang}^2 \frac{\beta}{2}}}{2} = \frac{(b-a)\cos^2 \frac{\varphi}{2}}{\cos \varphi}, \text{ si } \operatorname{tang} \varphi = \frac{2 \operatorname{tang} \frac{\beta}{2} \sqrt{ab}}{b-a}$$

$$d = a - b + c = \frac{a - b + \sqrt{(b-a)^2 + 4ab \operatorname{tang}^2 \frac{\beta}{2}}}{2} = \frac{(b-a)\sin^2 \frac{\varphi}{2}}{\cos \varphi}$$

Secundum §. 25. est $b \operatorname{tang} \frac{\beta}{2} \operatorname{tang} \frac{\gamma}{2} = d = \dfrac{a - b + \sqrt{(b-a)^2 + 4ab \operatorname{tang}^2 \frac{\beta}{2}}}{2}$

ergo $\operatorname{tang} \frac{\gamma}{2} = \cot \frac{\alpha}{2} = \dfrac{a - b + \sqrt{(b-a)^2 + 4ab \operatorname{tang}^2 \frac{\beta}{2}}}{2b \operatorname{tang} \frac{\beta}{2}} = \operatorname{tang} \frac{\varphi}{2} \sqrt{\frac{a}{b}}$

$J = \frac{1}{2}\,(ab + cd) \sin \beta$ et secundum §. 27. $cd = ab \operatorname{tang}^2 \frac{\beta}{2}$,

ergo $J = \frac{1}{2} ab (1 + \tang^2 \frac{\beta}{2}) \sin \beta = ab \tang \frac{\beta}{2}$

$$r = \frac{1}{2 \sin \beta} \sqrt{(b-a)^2 + 4ab \sin^2 \frac{\beta}{2}} = \frac{b-a}{2 \sin \beta \cos \psi}, \text{ si } \tang \psi = \frac{2 \sin \frac{\beta}{2} \sqrt{ab}}{b-a}$$

$$\varrho = \frac{J}{a+c} = \frac{2ab \tang \frac{\beta}{2}}{a+b+\sqrt{(b-a)^2 + 4ab \tang^2 \frac{\beta}{2}}} = \frac{\tang \frac{\beta}{2} [a+b-\sqrt{(b-a)^2 + 4ab \tang^2 \frac{\beta}{2}}]}{2 (1 - \tang^2 \frac{\beta}{2})}$$

$$= \frac{1}{4} \tang \beta [a+b-\sqrt{(b-a)^2 + 4ab \tang^2 \frac{\beta}{2}}] = \frac{1}{4} \tang \beta [a+b-\sqrt{(a+b)^2 - \frac{4ab \cos \beta}{\cos^2 \frac{\beta}{2}}}]$$

Igitur $\varrho = \frac{1}{2} (a+b) \tang \beta \sin^2 \frac{\omega}{2}$, si $\omega = \frac{2\sqrt{ab \cos \beta}}{(a+b) \cos \frac{\beta}{2}}$.

24) Problema 3. Datis 2 lateribus finitimis (a, b) et 1 angulo, qui ad unum laterum datorum adjacet (γ), quaerantur latera non data (c, d), anguli non dati (β, δ), area (J), radii circulorum (r, ϱ).

Solutio. Ex aequatione §. 27. $bc \tang^2 \frac{\gamma}{2} = ad = a(a+c-b)$ sequitur

$$c = \frac{a(a-b)}{b \tang^2 \frac{\gamma}{2} - a} = \frac{(a-b) \sin^2 \varphi}{\cos 2\varphi}, \quad \text{si } \cot \varphi = \tang \frac{\gamma}{2} \sqrt{\frac{b}{a}}$$

$$d = a - b + c = \frac{b(a-b) \tang^2 \frac{\gamma}{2}}{b \tang^2 \frac{\gamma}{2} - a} = \frac{(a-b) \cos^2 \varphi}{\cos 2\varphi}.$$

Secundum §. 25. est $b \tang \frac{\beta}{2} \tang \frac{\gamma}{2} = d = \frac{b(a-b) \tang^2 \frac{\gamma}{2}}{b \tang^2 \frac{\gamma}{2} - a}$,

$$\text{ergo } \tang \frac{\beta}{2} = \cot \frac{\delta}{2} = \frac{(a-b) \tang \frac{\gamma}{2}}{b \tang^2 \frac{\gamma}{2} - a} = \frac{(a-b) \tang \frac{\gamma}{2} \sin^2 \varphi}{a \cos 2\varphi}.$$

In problemate praecedenti inventum est $J = ab \sin \frac{\beta}{2}$,

$$\text{ergo } J = \frac{ab(a-b) \tang \frac{\gamma}{2}}{b \tang^2 \frac{\gamma}{2} - a} = \frac{b(a-b) \tang \frac{\gamma}{2} \sin^2 \varphi}{\cos 2\varphi}$$

$$r = \frac{1}{2\sin\gamma}\sqrt{(b-c)^2 + 4bc\sin^2\frac{\gamma}{2}}, \quad \text{sed } b - c = \frac{b^2\,\mathrm{tang}^2\frac{\gamma}{2} - a^2}{b\,\mathrm{tang}^2\frac{\gamma}{2} - a} \quad \text{et}$$

$$4bc\sin^2\frac{\gamma}{2} = \frac{4a^2 b\sin^2\frac{\gamma}{2} - 4ab^2\sin^2\frac{\gamma}{2}}{b\,\mathrm{tang}^2\frac{\gamma}{2} - a}; \quad \text{ergo}$$

$$r = \frac{1}{2\sin\gamma\,(b\,\mathrm{tang}^2\frac{\gamma}{2} - a)}\sqrt{(b\,\mathrm{tang}^2\frac{\gamma}{2} + a^2)\,(b^2\,\mathrm{tang}^2\frac{\gamma}{\alpha} + a^2 - 4ab\sin^2\frac{\gamma}{2})}$$

Secundum No. 10. est $\varrho = \dfrac{(a-b)\sin\frac{\alpha}{2}\sin\frac{\gamma}{2}}{\sin\frac{\gamma-\alpha}{2}}$; quum vero in his figuris $\alpha = 180^\circ - \gamma$,

$$\varrho = \frac{(a-b)\cos\frac{\gamma}{2}\sin\frac{\gamma}{2}}{\cos\gamma} = \frac{1}{2}(a-b)\,\mathrm{tang}\,\gamma.$$

25) Problema 4. Datis 2 lateribus oppositis (a, c) et 1 angulo (α), quaerantur latera non data (b, d), anguli non dati (β, δ), area (J), radii circulorum (r, ϱ).

Solutio. Quum sit secundum §. 27. $ad\,\mathrm{tang}^2\frac{\alpha}{2} = bc = c(c+a) - cd$,

$$d = \frac{c(a+c)}{a\,\mathrm{tang}^2\frac{\alpha}{2} + c} = (a+c)\cos^2\varphi, \quad \sin\mathrm{tang}\,\varphi = \mathrm{tang}\frac{\alpha}{2}\sqrt{\frac{a}{c}},$$

$$b = a + c - d = \frac{a(a+c)\,\mathrm{tang}^2\frac{\alpha}{2}}{a\,\mathrm{tang}^2\frac{\alpha}{2} + c} = (a+c)\sin^2\varphi$$

Ex §. 25. sequitur $\quad \mathrm{tang}\frac{\beta}{2} = \cot\frac{\delta}{2} = \frac{c}{a}\cot\frac{\alpha}{2}$.

Secundum No. 12. est $J = \dfrac{a(a+c)\sin\frac{\alpha}{2}\sin\frac{\beta}{2}}{\sin\frac{\alpha+\beta}{2}} = \dfrac{a(a+c)}{\cot\frac{\beta}{2} + \cot\frac{\alpha}{2}}$, ergo

$$J = \frac{ac\,(a+c)\cot\frac{\alpha}{2}}{a + c\cot^2\frac{\alpha}{2}} = a(a+c)\,\mathrm{tang}\frac{\alpha}{2}\cos^2\varphi.$$

$$r = \frac{1}{2\sin\alpha}\sqrt{(a-d)^2 + 4ad\sin^2\frac{\alpha}{2}}. \quad \text{Sed } a - d = \frac{a^2\,\mathrm{tang}^2\frac{\alpha}{2} - c^2}{a\,\mathrm{tang}^2\frac{\alpha}{2} + c}; \quad \text{igitur}$$

$$r = \frac{1}{2 \sin \alpha \left(a^2 \operatorname{tang}^2 \frac{\alpha}{2} + c\right)} \sqrt{\left(a^2 \operatorname{tang}^2 \frac{\alpha}{2} + c^2\right)\left(a^2 \operatorname{tang}^2 \frac{\alpha}{2} + c^2 + 4ac \sin^2 \frac{\alpha}{2}\right)}$$

$$\varrho = \frac{J}{a + c} = \frac{ac \cot \frac{\alpha}{2}}{a + c \cot^2 \frac{\alpha}{2}} = a \operatorname{tang} \frac{\alpha}{2} \cos^2 \varphi .$$

26) **Problema 5.** Datis 1 latere (a) et 2 angulis finitimis (α, δ), quaerantur latera non data (b, c, d), area (J), radii circulorum (r. ϱ).

Solutio. Ex §. 23. sequitur $d = \dfrac{a \sin \frac{\alpha + \delta}{2}}{\operatorname{tang} \frac{\delta}{2} \cos \frac{\alpha - \delta}{2}}$, $b = \dfrac{a \operatorname{tang} \frac{\alpha}{2} \sin \frac{\alpha + \delta}{2}}{\cos \frac{\alpha - \delta}{2}}$

et ex §. 26. $c = a \operatorname{tang} \frac{\alpha}{2} \cot \frac{\delta}{2}$.

$$\varrho = \frac{a \sin \frac{\alpha}{2} \sin \frac{\beta}{2}}{\sin \frac{\alpha + \beta}{2}} = \frac{a \sin \frac{\alpha}{2} \cos \frac{\delta}{2}}{\cos \frac{\alpha - \delta}{2}}$$

$$J = (a + c)\varrho = a\left(1 + \operatorname{tang} \frac{\alpha}{2} \cot \frac{\delta}{2}\right) \frac{a \sin \frac{\alpha}{2} \cos \frac{\delta}{2}}{\cos \frac{\alpha - \delta}{2}}, \text{ i.e. } J = \frac{a^2 \sin \frac{\alpha + \delta}{2} \operatorname{tang} \frac{\alpha}{2} \cot \frac{\delta}{2}}{\cos \frac{\alpha - \delta}{2}}$$

$$r = \frac{1}{2 \sin \alpha} \sqrt{(d - a)^2 + 4ad \sin^2 \frac{\alpha}{2}}; \text{ est vero}$$

$$d - a = \frac{a\left(\sin \frac{\alpha + \delta}{2} - \operatorname{tang} \frac{\delta}{2} \cos \frac{\alpha - \delta}{2}\right)}{\operatorname{tang} \frac{\delta}{2} \cos \frac{\alpha - \delta}{2}} = \frac{a \sin \frac{\alpha}{2} \cos \delta}{\sin \frac{\delta}{2} \cos \frac{\alpha - \delta}{2}} \text{ et}$$

$$(d - a)^2 + 4ad \sin^2 \frac{\alpha}{2} = a^2 \sin^2 \frac{\alpha}{2} \left(\frac{\cos^2 \delta}{\sin^2 \frac{\delta}{2} \cos^2 \frac{\alpha - \delta}{2}} + \frac{4 \cos \frac{\delta}{2} \sin \frac{\alpha + \delta}{2}}{\sin \frac{\delta}{2} \cos \frac{\alpha - \delta}{2}}\right)$$

$$= \frac{a^2 \sin^2 \frac{\alpha}{2}}{\sin^2 \frac{\delta}{2} \cos^2 \frac{\alpha - \delta}{2}} \left(\cos^2 \delta + 4 \sin \frac{\delta}{2} \cos \frac{\delta}{2} \sin \frac{\alpha + \delta}{2} \cos \frac{\alpha - \delta}{2}\right) = \frac{a^2 \sin^2 \frac{\alpha}{2}}{\sin^2 \frac{\delta}{2} \cos^2 \frac{\alpha - \delta}{2}} (1 + \sin \alpha \sin \delta)$$

$$\text{Ergo } r = \frac{a}{4 \cos \frac{\alpha}{2} \sin \frac{\delta}{2} \cos \frac{\alpha - \delta}{2}} \sqrt{1 + \sin \alpha \sin \delta} .$$

Kolberg.